Condensation of gauge interacting mass-less fermions

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A single mass-less fermionic field with an abelian U(1) gauge interaction (electrodynamics of a mass-less Dirac fermion) is studied by a variational method. Even without the insertion of any extra interaction the vacuum is shown to be unstable towards a particle-antiparticle condensate. The single particle excitations do acquire a mass and behave as massive Fermi particles. An explicit low-energy gap equation has been derived and numerically solved. Some consequences of condensation and mass generation are discussed in the framework of the standard model.

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I. INTRODUCTION

One of the most outstanding problems in the Standard Model (SM) of strong and electroweak interactions is the very nature of the Higgs boson. That heavy scalar particle has not been found yet, but is required by the theory in order to give a mass to all the fermions through the standard symmetry breaking mechanism. The nature of the scalar field is not established and several composite models have been proposed. Top-quark condensation [1] is one of the most interesting and economical mechanisms as it does not require the existence of any new particle. The basic idea goes back to Nambu and Jona-Lasinio's [2] work on fermion condensation which appeared in 1961, just four years later than the celebrated paper on superconductivity by Bardeen, Cooper and Schrieffer (BCS) [3]. The formal analogy between the single particle excitation energies of a superconductor $\epsilon_k = \sqrt{\mathbf{k}^2 + \Delta^2}$ and the relativistic mass-shell equation (vacuum excitation energies) for a massive particle is striking and would suggest that the mass is just the gap Δ which opens in some condensation mechanism. However, all Top-quark condensation models require the existence of an *ad hoc* four-fermion interaction which must be added to the standard lagrangian. Then the SM is recovered as a low energy effective theory with the scalar Higgs field describing the condensate.

In this paper we discuss the problem of fermion condensation without adding any four-fermion attraction. In fact by a variational method we show that the standard gauge interactions may give rise to condensation of the original massless fermions. We address the problem by use of a simplified toy model consisting of a single massless fermionic field with an abelian $U(1)$ gauge interaction (electrodynamics of a massless Dirac fermion). In this framework condensation may have several important consequences: first of all the SM consists of gauge interacting massless fermions, and their *spontaneous* condensation could spoil the standard symmetry breaking and mass generation mechanisms which are supposed to be due to the interaction with the scalar field. Moreover, a *spontaneous* condensation could replace the standard mechanism without having to insert any extra field or interaction.

Again the idea that gauge interactions could give rise to condensation of massless fermions is not new, and comes from condensed matter. Ten years after the BCS paper [3], Jerome, Rice and Kohn [4] predicted the existence of an *excitonic insulator* in gapless semiconductors as a consequence of the condensation of electron-hole pairs bound by the coulomb interaction. Since the pair (the exciton) does not carry any charge, the $U(1)$ symmetry is unbroken, and condensation does not give rise to any superconductive property. However a gap opens in the spectrum of single particle excitation energies, and the quasi-particles do acquire a mass (the system is predicted to be an insulator).

For a generic charged massless fermionic field we predict the occurrence of a spontaneous condensation for any weak long range coupling: as single particle excitation energies are gapless in the original vacuum, any weak long range attraction would drive the condensation of particle-antiparticle pairs. In the new vacuum the single particle excitation energies (the physical particles) acquire a finite mass which obviously depends on the only energy scale of the problem: the energy cut-off required in order to regularize the theory.

In our toy model the mass is very small compared to fermionic masses, unless the cut-off is allowed to take huge values. Thus below the grand unified scale ($\sim 10^{16}$ GeV) spontaneous condensation should not spoil the SM symmetry breaking mechanism. However a mass as large as the electron rest mass can be recovered by a cut-off approaching the Landau singularity point. In the light of these findings, the full non-abelian theory should be studied in order to achieve a quantitative estimate of the effect which could be relevant in the context of top-quark condensation. On

the other hand, a small mass could also arise for neutrinos, as a consequence of weak interactions. We notice that in general this spontaneous condensation would break chiral invariance while leaving the $U(1)$ symmetry unbroken. Such open problems, while motivating our work, go beyond the aims of the present paper which only deals with the existence of spontaneous condensation and is organized as follows: in the next section the toy model is defined and the variational method is described; in section III a gap equation is obtained and details of the derivation are presented; section IV contains a full discussion of the analytical results and comments on their relevance; a low-energy gap equation is derived in section V where a more formal proof is given of the existence of a non-trivial solution. In that section the problem of mass generation is discussed by numerical solution of the low-energy gap equation.

II. THE MODEL

Let us consider the following toy-model lagrangian:

$$\mathcal{L} = -\bar{\Psi}\left(\gamma^\mu \partial_\mu + ie\gamma^\mu A_\mu\right)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1}$$

It containes a Dirac mass-less $S = \frac{1}{2}$ fermionic field Ψ with a $U(1)$ (e.m.) gauge interaction field A_μ, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The electric current is

$$J^\mu = \frac{\partial \mathcal{L}}{\partial A_\mu} = -ie\bar{\Psi}\gamma^\mu\Psi \tag{2}$$

In Coulomb gauge the hamiltonian reads

$$H = H_0 + V_c + V_i \tag{3}$$

where H_0 is the free particle hamiltonian

$$H_0 = \int d^3p \sum_\sigma |\mathbf{p}| a^\dagger(\mathbf{p},\sigma)a(\mathbf{p},\sigma) + \int d^3p \sum_\sigma |\mathbf{p}| \left[\alpha^\dagger(\mathbf{p},\sigma)\alpha(\mathbf{p},\sigma) + \beta^\dagger(\mathbf{p},\sigma)\beta(\mathbf{p},\sigma)\right] \tag{4}$$

(here $a, a^\dagger$ are photon annihilation,creation operators, while $\alpha, \alpha^\dagger$ and $\beta, \beta^\dagger$ are particle and anti-particle operators for the mass-less fermions), V_c is the Coulomb interaction

$$V_c = \frac{1}{2}\int d^3x \int d^3y \frac{J^0(x)J^0(y)}{4\pi|\mathbf{x}-\mathbf{y}|} \tag{5}$$

and V_i is the interaction term

$$V_i = -\int d^3x \mathbf{J}(x)\mathbf{A}(x). \tag{6}$$

Let us take as a trial vacuum the BCS-like vacuum discussed by Nambu and Jona-Lasinio [2]

$$|\Phi\rangle = \prod_{\mathbf{k},\sigma}\left[h_k + g_k \alpha^\dagger(\mathbf{k},\sigma)\beta^\dagger(-\mathbf{k},\sigma)\right]|0\rangle \tag{7}$$

where $|0\rangle$ is the vacuum annihilated by a, α, β. For $g_k = 0$ this trial vacuum contains the trivial vacuum $|0\rangle$. For $g_k \neq 0$ the trial vacuum is characterized by pair correlation: each pair has zero charge, zero momentum and zero spin (σ denotes helicity). Thus, at variance with superconductivity, we expect e.m. $U(1)$ gauge invariance to be unbroken (Eventually $SU(2)_L$ could be broken since the pair carries two units of chirality).

Next let us evaluate the ground state energy

$$E = \langle\Phi|H|\Phi\rangle \tag{8}$$

The average value of V_i vanishes (it is linear in a and $a^\dagger$), and we only need

$$E = \langle\Phi|H_0|\Phi\rangle + \langle\Phi|V_c|\Phi\rangle \tag{9}$$

Actually V_c is a four-fermion correlation interaction:

$$V_c = \frac{1}{2}\int \frac{d^3q}{(2\pi)^3}\frac{J^0(\mathbf{q})J^0(-\mathbf{q})}{\mathbf{q}^2} \tag{10}$$

where $J^0(\mathbf{q})$ is the Fourier transform

$$J^0(\mathbf{q}) = \int d^3x e^{-i\mathbf{qx}}\left[-e\Psi^\dagger(x)\Psi(x)\right] \tag{11}$$

and the fermion field $\Psi(x)$ is given as usual by

$$\Psi(x) = \frac{1}{(2\pi)^{3/2}}\int d^3p\sum_\sigma\left[e^{i\mathbf{px}}u(\mathbf{p},\sigma)\alpha(\mathbf{p},\sigma) + e^{-i\mathbf{px}}v(\mathbf{p},\sigma)\beta^\dagger(\mathbf{p},\sigma)\right]. \tag{12}$$

Insertion of Eq.(12) and (11) in Eq.(10) yields a four-fermion interaction. The average energy E in Eq.(9) may be evaluated by use of the properties of the spinors u and v (spin sums and traces).

The coefficients g_k and h_k are then regarded as variational parameters, with the normalization constraint

$$|g_k|^2 + |h_k|^2 = 1 \tag{13}$$

and the true vacuum (ground state) is recovered by differentiating Eq.(9).

III. THE GAP EQUATION

The explicit evaluation of Eq.(9) is straightforward. The average value of V_c reads

$$\begin{aligned}\langle\Phi|V_c|\Phi\rangle = e^2\Omega\sum\int \frac{d^3pd^3p'd^3q}{(2\pi)^3\mathbf{q}^2}\times \\ \times\langle\Phi|\eta_i^\dagger(\mathbf{p}-\mathbf{q},\sigma)\eta_j(\mathbf{p},\tau)\eta_k^\dagger(\mathbf{p}'+\mathbf{q},\sigma')\eta_l(\mathbf{p}',\tau')|\Phi\rangle\times \\ \times S_i^\dagger(\mathbf{p}-\mathbf{q},\sigma)S_j(\mathbf{p},\tau)S_k^\dagger(\mathbf{p}'+\mathbf{q},\sigma')S_l(\mathbf{p}',\tau')\end{aligned} \tag{14}$$

where Ω is the space volume and the sum runs over helicities σ,σ',τ,τ' and over $ijkl = 1, 2$, with the notation

$$\begin{aligned}\eta_1^\dagger(\mathbf{p},\sigma) &= \alpha^\dagger(\mathbf{p},\sigma)\\ \eta_2^\dagger(\mathbf{p},\sigma) &= \beta(-\mathbf{p},\sigma)\\ S_1(\mathbf{p},\sigma) &= u(\mathbf{p},\sigma)\\ S_2(\mathbf{p},\sigma) &= v(-\mathbf{p},\sigma).\end{aligned} \tag{15}$$

This average vanishes unless any $\eta^\dagger(\mathbf{p},\sigma)$ is joined by a $\eta(\mathbf{p},\sigma)$ with the same arguments: as the trial state $|\Phi\rangle$ only contains pair correlations the action of any creation operator must be followed by the annihilation of the same particle or by the creation of the paired antiparticle. We have three cases: i) $\mathbf{q} = 0$, $\sigma' = \tau'$, $\sigma = \tau$ (with $\mathbf{p} \neq \mathbf{p}'$ or $\sigma \neq \sigma'$); ii) $\mathbf{q} = 0$, $\mathbf{p} = \mathbf{p}'$, $\sigma = \sigma' = \tau = \tau'$; iii)$\mathbf{p} - \mathbf{p}' = \mathbf{q}$, $\sigma = \tau'$, $\tau = \sigma'$ (with $\mathbf{q} \neq 0$ or $\sigma \neq \sigma'$). Both cases i) and ii) only contribute constant terms (in the sense that they do not depend on g_k) and may be dropped as they give the same energy contribution in the trivial vacuum $|0\rangle$. For the case iii) the sum over helicities yields

$$\sum_{\sigma\tau} S_i^\dagger(\mathbf{p}',\sigma)S_j(\mathbf{p},\tau)S_k^\dagger(\mathbf{p},\tau)S_l(\mathbf{p}',\sigma) = Tr\left[A_{jk}(\mathbf{p})A_{li}(\mathbf{p}')\right] \tag{16}$$

where for each pair (ij) the matrix A_{ij} is

$$A_{ij}(\mathbf{p}) = \sum_\sigma S_i(\mathbf{p},\sigma)S_j^\dagger(\mathbf{p},\sigma). \tag{17}$$

The non-vanishing contributions are

$$Tr\left[A_{11}(\mathbf{p})A_{11}(\mathbf{p}')\right] = 1 + \frac{\mathbf{p}\cdot\mathbf{p}'}{|\mathbf{p}||\mathbf{p}'|}$$

$$
\begin{aligned}
Tr\left[A_{11}(\mathbf{p})A_{22}(\mathbf{p}')\right] &= 1-\frac{\mathbf{p}\cdot\mathbf{p}'}{|\mathbf{p}||\mathbf{p}'|}\\
Tr\left[A_{12}(\mathbf{p})A_{12}(\mathbf{p}')\right] = Tr\left[A_{21}(\mathbf{p})A_{21}(\mathbf{p}')\right] &= 1-\frac{\mathbf{p}\cdot\mathbf{p}'}{|\mathbf{p}||\mathbf{p}'|}\\
Tr\left[A_{12}(\mathbf{p})A_{21}(\mathbf{p}')\right] = Tr\left[A_{21}(\mathbf{p})A_{12}(\mathbf{p}')\right] &= 1+\frac{\mathbf{p}\cdot\mathbf{p}'}{|\mathbf{p}||\mathbf{p}'|}.
\end{aligned}
\tag{18}
$$

Inserting the coefficients (16) for the respective averages $\langle\eta_i^\dagger\eta_j\eta_k^\dagger\eta_l\rangle$ we obtain

$$
\langle\Phi|V_c|\Phi\rangle = E_{EXC} + E_{BCS} \tag{19}
$$

where

$$
E_{EXC} = 2e^2\Omega\int\frac{d^3p\,d^3k}{(2\pi)^6}\frac{|g_p|^2|h_k|^2\mathbf{p}\cdot\mathbf{k}}{|\mathbf{p}-\mathbf{k}|^2|\mathbf{p}||\mathbf{k}|} \tag{20}
$$

$$
E_{BCS} = -2e^2\Omega\int\frac{d^3p\,d^3k}{(2\pi)^6}\frac{h_p g_p h_k^* g_k^*}{|\mathbf{p}-\mathbf{k}|^2} \tag{21}
$$

The average of H_0 is trivial

$$
\langle\Phi|H_0|\Phi\rangle = E_0 = 4\Omega\int\frac{d^3p}{(2\pi)^3}|g_p|^2|\mathbf{p}| \tag{22}
$$

and the total energy reads

$$
E = E_0 + E_{BCS} + E_{EXC} \tag{23}
$$

While E_{BCS} is the usual BCS pairing energy, the term E_{EXC} survives from a partial compensation of the exchange energies in the particle-antiparticle condensate. As the effects of this exchange term can be dealt with by standard perturbative renormalization of parameters, we neglect this term in order to simplify the gap equation. In the appendix we show that the inclusion of the exchange term would give rise to a charge renormalization which is equivalent to the standard perturbative renormalization up to first order. In the following discussion we assume that both charge and mass are the physical renormalized values in order to incorporate the effects of the exchange energy and of the other interactions which have been neglected in this simple toy model. Conversely the term E_{BCS} cannot be dealt with by standard perturbation theory as it makes the trivial vacuum unstable. As usual we attempt a variational estimate of the best g_p value by differentiating the total energy $E = E_0 + E_{BCS}$.

According to the normalization condition (13) we denote

$$
\begin{aligned}
g_p &= \sin\theta_p\\
h_p &= \cos\theta_p
\end{aligned}
\tag{24}
$$

and differentiate the energy

$$
\frac{1}{\Omega}\frac{\delta E}{\delta\theta_p} = 4|\mathbf{p}|\sin 2\theta_p - 2e^2\cos 2\theta_p\int\frac{d^3k}{(2\pi)^3}\frac{\sin 2\theta_k}{|\mathbf{p}-\mathbf{k}|^2} = 0. \tag{25}
$$

This can be written as a standard gap equation

$$
\Delta_p = \frac{e^2}{2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{|\mathbf{p}-\mathbf{k}|^2}\frac{\Delta_k}{\sqrt{\mathbf{k}^2+\Delta_k^2}} \tag{26}
$$

where the gap function Δ_k is defined according to

$$
\frac{\Delta_p}{|\mathbf{p}|} = \frac{\sin 2\theta_p}{\cos 2\theta_p} = \frac{e^2}{2|\mathbf{p}|}\int\frac{d^3k}{(2\pi)^3}\frac{\sin 2\theta_k}{|\mathbf{p}-\mathbf{k}|^2}. \tag{27}
$$

Eq.(26) is the gap equation we would have expected from the beginning. The trivial vacuum $|0\rangle$ is given by the solution $\Delta_k = 0$ which is not the ground state as an other unconventional solution can be always found for any strength of the coupling constant.

IV. CONDENSATION AND FERMION MASSES

The non trivial solution of the gap equation (26) describes a particle-antiparticle condensate. The single particle excitations are the "physical" particles and are characterized by the energy spectrum

$$\varepsilon_k = \sqrt{\mathbf{k}^2 + \Delta_k}. \tag{28}$$

Thus the single particle excitations behave like massive fermions with a mass $M = \Delta_0$. The IR behaviour of Eq.(26) requires the existence of a non vanishing mass M. In fact at low energy, replacing $\Delta_k \approx M$ the gap equation reads

$$\frac{2}{e^2} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\mathbf{k}^2\sqrt{\mathbf{k}^2 + M^2}} \tag{29}$$

and the logarithmic divergence for $\mathbf{k} \to 0$ ensures that the mass M is not vanishing. The exact value depends on the high energy behaviour of the function Δ_k: the UV convergence of the integrals requires that Δ_k should be vanishing at large energies. We may estimate M by the ansatz $\Delta_k = 0$ for $|\mathbf{k}| > \Lambda$ which is equivalent to the insertion of a cut-off Λ in Eq.(29) thus obtaining

$$M \approx 2\Lambda e^{-4\pi^2/e^2}. \tag{30}$$

Thus the variational method shows that a non-vanishing mass M is always present for any weak coupling e^2. As for superconductivity this result cannot be obtained by any perturbative expansion in e^2 starting from the trivial vacuum $|0\rangle$. Of course the mass M depends on the unique length scale of the model which is the energy cut-off Λ. Moreover the mass M is very small unless the cut-off Λ is supposed to be really huge. The scale Λ would represent an intrinsic limit of the simple electrodynamics of a charged fermion. It is remarkable that, assuming for M the phenomenological value of the electron mass, the cut-off Λ reaches the large value

$$\Lambda \approx \frac{M}{2} e^{4\pi^2/e^2} \tag{31}$$

close to the Landau singularity

$$\Lambda_{Landau} = M e^{6\pi^2/e^2}, \tag{32}$$

but still smaller.

We notice that for any weak coupling, the occurrence of particle-antiparticle condensation and the opening of a gap are the natural consequence of two important aspects: i) the vanishing of mass in the trivial vacuum; ii) the long range behaviour of the Coulomb interaction.

The trivial vacuum for mass-less fermions is unstable because the creation of a particle-antiparticle pair does not require any energy, while the particle and the antiparticle attract each other: thus the creation energy may become negative. This is known to be the case for semiconductors when the gap is vanishing. However, at variance with condensed matter, the vanishing of the gap is not by itself a sufficient condition for determining the vacuum instability. In superconductors the integration over $\mathbf{k}$ is carried across the Fermi level, where $|\mathbf{k}| = k_F$ is a finite Fermi vector, and the measure only gives a simple dk factor which is not enough for the IR convergence. The opening of the gap is then necessary as otherwise in the gap equation the integral would diverge logarithmically. Conversely here the integration reaches the $\mathbf{k} = 0$ point and the measure gives a $\mathbf{k}^2 dk$ term which would be enough for the IR convergence of the integral in Eq.(26) were it not for the extra IR divergence of the Coulomb interaction $e^2/\mathbf{k}^2$. Actually it is well known that for any short-range interaction condensation only takes place if the coupling is strong enough [2]. The diverging $1/\mathbf{k}^2$ behaviour of the Coulomb interaction here cancels the extra measure factor $\mathbf{k}^2$ thus restoring the IR logarithmic divergence in the gap equation: the long range behaviour of gauge interactions is a key factor for the condensation of mass-less fermions.

V. MASS GENERATION AND LOW-ENERGY GAP EQUATION

In the light of the present study we believe it correct to say that fermion condensation does not spoil the SM mechanism of symmetry breaking, as far as fermions do acquire a mass by interaction with the Higgs field or by some other effect. However fermion condensation could itself be a candidate to such symmetry breaking description

with the condensate playing the role of the scalar field. In that framework any further discussion on mass generation would require a numerical solution of the gap equation Eq.(26), at least in the low-energy domain. Unfortunately the huge ratio Λ/M rules out a direct numerical evaluation, as the gap turns out to be too small for any viable cut-off Λ. In fact any numerical attempt to solve Eq.(26) would question the existence of any non-trivial solution. However a non-vanishing numerical solution is easily found by iteration as long as the coupling parameter e^2 is taken large enough to keep the ratio Λ/M at a numerically tractable value according to Eq.(30). From the existence of a non-trivial solution at large coupling we may prove that the gap must remain non-vanishing for any weak coupling. In fact, should the gap vanish at a critical coupling $e^2 = e_c^2$, we could expand the gap equation Eq.(26) in powers of Δ_k at that critical point. The first order approximation is easily obtained by inserting $\Delta_k = 0$ in the square root, and holds for a vanishing gap. Thus at the critical point the gap equation can be replaced by the eigenvalue problem of a linear integral operator. The existence of a non-trivial solution for $e^2 > e_c^2$ is equivalent to say that $1/e_c^2$ is the larger eigenvalue of the linear integral operator. But the integral operator is not bounded as the kernel diverges for $p \to 0$, $k \to 0$, and thus $e_c^2 = 0$. That means the non-trivial solution becomes very small for a weak coupling but does not vanish as long as the coupling is $e^2 > 0$.

A more tractable low-energy gap equation can be derived for the realistic weak coupling limit $e^2/(4\pi) = \alpha \approx 1/137$ of quantum electrodynamics. Let us consider the arbitrary intermediate scale μ, assuming $\Delta_p \ll \mu$ for any p, but $\mu \ll \Lambda$. By integrating over angles Eq.(30) reads

$$\Delta_p = \frac{\alpha}{2\pi} \int_0^\mu dk\ \frac{k}{p} \ln\left|\frac{p+k}{p-k}\right| \frac{\Delta_k}{\sqrt{k^2+\Delta_k^2}} + \frac{\alpha}{2\pi} \int_\mu^\Lambda dk\ \frac{k}{p} \ln\left|\frac{p+k}{p-k}\right| \frac{\Delta_k}{\sqrt{k^2+\Delta_k^2}} \tag{33}$$

In the low energy domain $p \ll \mu$ the second integral may be approximated by

$$\frac{\alpha}{2\pi} \int_\mu^\Lambda dk\ \frac{k}{p} \ln\left|\frac{p+k}{p-k}\right| \frac{\Delta_k}{\sqrt{k^2+\Delta_k^2}} = \frac{\alpha}{\pi} \int_\mu^\Lambda dk\ \frac{\Delta_k}{k} + \mathcal{O}\left(p^2/\mu^2\right) \tag{34}$$

since $p \ll \mu < k$ and $\Delta_k \ll \mu < k$ so that $\sqrt{\Delta_k^2 + k^2} \approx k$. For small p we may drop the dependence on p and write

$$\Delta_p = M_\mu + \frac{\alpha}{2\pi} \int_0^\mu dk\ \frac{k}{p} \ln\left|\frac{p+k}{p-k}\right| \frac{\Delta_k}{\sqrt{k^2+\Delta_k^2}} \tag{35}$$

where any dependence on Λ is now in the renormalized mass M_μ which is defined at the scale μ as

$$M_\mu = \frac{\alpha}{\pi} \int_\mu^\Lambda dk\ \frac{\Delta_k}{k}. \tag{36}$$

Eq.(35) is an approximate low-energy gap equation for Δ_k in the range $k < \mu$. The high energy behaviour of Δ_k would only be required in order to evaluate the renormalized mass constant M_μ. However, by a proper choice of Λ, M_μ can take any chosen value and can be regarded as a free parameter incorporating any dependence on the cut-off. Moreover M_μ must be close to the phenomenological mass as the integral in Eq.(35) yields a very small contribution. The low-energy gap equation Eq.(35) always has a solution which can be easily evaluated by numerical iteration. The numerical solution is shown in Fig.1 for the phenomenological coupling $\alpha = 1/137$ and for two different scales: a small scale $\mu = 1.5 M_\mu$, and a large scale $\mu = 220 M_\mu$. For small p the gap Δ_p is almost constant, with a decrease of less than 0.15% in the range $0 < p < 2M_\mu$ of the large scale solution. Thus we may regard the $p \to 0$ limit of Δ_p as the "physical" mass $M = \Delta_0$.

A perturbative iterative solution in powers of the coupling α may be written for the low-energy gap equation by inserting $\Delta_p = M_\mu + \mathcal{O}(\alpha)$ in the right hand side of Eq.(35), yielding in the limit $p \to 0$

$$M \approx M_\mu + \frac{\alpha}{\pi} M_\mu \ln\left(\frac{\mu}{M_\mu}\right) + \mathcal{O}\left(\alpha^2\right) \tag{37}$$

This first order mass shift turns out to be just 2/3 times the standard perturbative self-energy contribution at the energy scale μ.

While fermion condensation would be the simplest mass generation mechanism, it would also provide the existence of a condensate playing the role of the Higgs field. Moreover, as pairs carry a unit of helicity, any heli-magnetic solution would give a simple picture of left-right symmetry breaking. Quite interesting, as weak interactions are long ranged before symmetry breaking, even neutrinos could undergo condensation and would acquire a very small mass.

All these outstanding problems call for a detailed study of the full non-abelian gauge group which goes beyond the aim of the present paper.

APPENDIX A: EXCHANGE ENERGY AND CHARGE RENORMALIZATION

Inclusion of the exchange term E_{EXC} Eq. (20) in the total energy would only give rise to a more complicated gap equation which can be cast again in the shape of Eqs.(26) and (29) by charge renormalization. Insertion of Eq.(24) and differentiation yield the following gap equation

$$\Delta_p = \frac{e^2}{2(1+I_p)} \int \frac{d^3k}{(2\pi)^3} \frac{1}{|\mathbf{p}-\mathbf{k}|^2} \frac{\Delta_k}{\sqrt{\mathbf{k}^2+\Delta_k^2}} \tag{A1}$$

where the gap function Δ_k is defined according to

$$\frac{\Delta_p}{|\mathbf{p}|} = \frac{\sin 2\theta_p}{\cos 2\theta_p} = \frac{e^2}{2|\mathbf{p}|(1+I_p)} \int \frac{d^3k}{(2\pi)^3} \frac{\sin 2\theta_k}{|\mathbf{p}-\mathbf{k}|^2}. \tag{A2}$$

and the integral I_p is

$$I_p = \frac{e^2}{2|\mathbf{p}|} \int \frac{d^3k}{(2\pi)^3} \frac{(\mathbf{k}\cdot\mathbf{p})\cos 2\theta_k}{|\mathbf{p}||\mathbf{k}||\mathbf{p}-\mathbf{k}|^2}. \tag{A3}$$

For small energies

$$I_p = e^2 A + \mathcal{O}(p) \tag{A4}$$

where the constant A is given by the integral

$$A = \frac{1}{6\pi^2} \int_0^\Lambda \frac{dk}{\sqrt{k^2+\Delta_k^2}}. \tag{A5}$$

which is regularized by a cut-off Λ. Thus for $M = \Delta_0$ Eq.(29) is recovered as

$$\frac{2}{Z_3 e^2} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\mathbf{k}^2\sqrt{\mathbf{k}^2+M^2}} \tag{A6}$$

where the renormalization constant Z_3 reads

$$Z_3 = \frac{1}{1+e^2 A} \tag{A7}$$

and up to order e^2, inserting Eq.(A5), we obtain

$$Z_3 \approx 1 - \frac{e^2}{6\pi^2} \ln(\Lambda/M) \tag{A8}$$

which is exactly the standard first order perturbative result for Z_3. As shown by Eq.(A6) all the discussions on Eq.(29) still hold provided that the renormalized coupling $Z_3 e^2$ is substituted for the bare coupling e^2.

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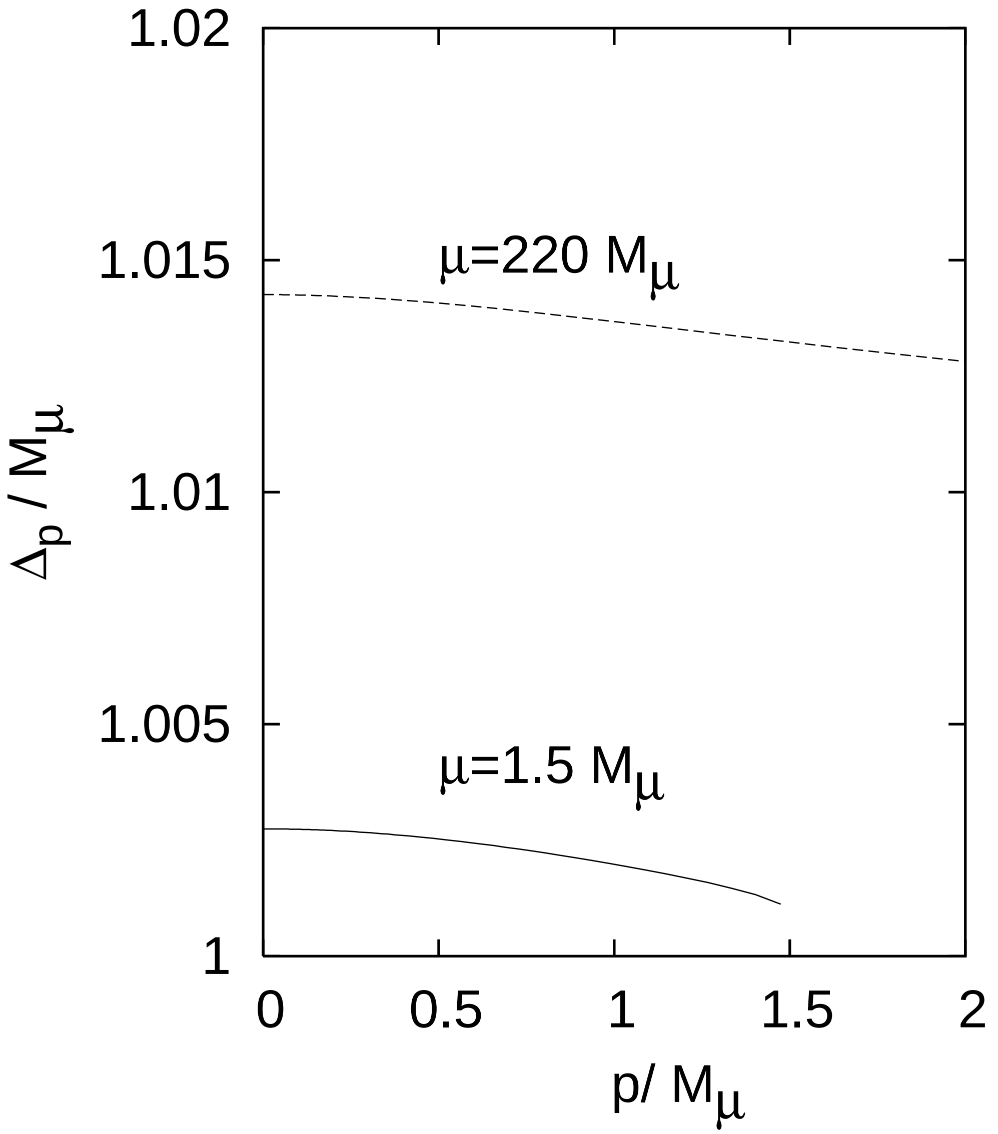
FIG. 1. Numerical solution of the low-energy gap equation Eq.(35) for $\alpha = 1/137$. The gap Δ_p is reported as a function of p in units of the mass constant M_μ, and for two different energy scales $\mu = 1.5 M_\mu$ (solid line) and $\mu = 220 M_\mu$ (dashed line).



1.02
1.015
1.01
1.005
1
0
0.5
1
1.5
2
Δp / Mμ
p/ Mμ
μ=220 Mμ
μ=1.5 Mμ